with previously published work was performed and the results were found to be in excellent agreement. It is hoped that the present work will serve as a stimulus for needed experimental work, which appears to be lacking at present.

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Green's Function to Determine Temperature Distribution in a Semitransparent Thermal Barrier Coating

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Nomenclature

a_λ = spectral absorption coefficient of semitransparent coating, m^{-1}
$e_{\lambda b}$ = blackbody spectral energy in vacuum, $W/m^2\ \mu m$; $\tilde{e}_{\lambda b} = e_{\lambda b}/\sigma T_{g1}^4$
F_{uj}, F_{lj} = fraction of blackbody energy from $\lambda = 0$ to the upper or lower limit of a band
G_λ = spectral flux quantity in two-flux method, $W/m^2\ \mu m$; $\tilde{G}_\lambda = G_\lambda/\sigma T_{g1}^4$
g_λ = Green's function for $\tilde{G}_\lambda(X)$ in coating
h_1, h_2 = convective heat transfer coefficients at boundaries, W/m^2 K; $H = h/\sigma T_{g1}^3$
K_λ = spectral extinction coefficient of semitransparent coating, $a_\lambda + \sigma_{s\lambda}$, m^{-1}
k_c, k_m = thermal conductivity (coating, metal), W/m K; $N_c = k_c/\sigma T_{g1}^3\delta_c$
m_λ = quantity $K_\lambda\delta_c[3(1 - \Omega_\lambda)]^{1/2}$
n = refractive index of semitransparent coating
$P_{i\lambda}$, $P_{m\lambda}$ = quantities $2R_i/3K_\lambda\delta_c$ and $2R_m/3K_\lambda\delta_c$
q = heat flux, W/m^2; $\tilde{q} = q/\sigma T_{g1}^4$
q_{tot} = total heat flux in x direction, W/m^2
R_i, R_o = the quantities, $(1 + \rho_i)/(1 - \rho_i)$ and $(1 - \rho_o)/(1 - \rho_i)$
R_m = the quantity $(2/\varepsilon_m) - 1$
T = absolute temperature, K; $t = T/T_{g1}$
T_{g1}, T_{g2} = gas temperatures on hot and cold sides of composite, Fig. 1, K
T_{s1} = blackbody temperature of surroundings outside of coating, K
x = coordinate in composite, Fig. 1, m; $X = x/\delta_c$
δ_c, δ_m = thicknesses of coating and metal, m
ε_m = emissivity of metal
κ_j = optical thickness of coating in jth wavelength band, $K_j\delta_c$
λ = wavelength of radiation, μm
ρ = diffuse reflectivity of interface from Fresnel equations using n value
σ = Stefan–Boltzmann constant, $W/m^2\ K^4$
$\sigma_{s\lambda}$ = scattering coefficient in semitransparent coating, m^{-1}; $\Omega_\lambda = \sigma_{s\lambda}/K_\lambda$

Subscripts

c, m = coating and metal
j, J = in jth or Jth spectral band
l, u = lower and upper limits of the jth spectral band
o, i = outside and inside interfaces at the external boundary of coating, Fig. 1
r = radiative
λ = spectral quantity

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Introduction

THERMAL barrier coatings on turbine vanes and rotating blades are important for reducing metal temperatures in current and advanced aircraft engines. A common coating material is zirconia that is partially transparent to thermal radiation.[1,2] Radiation will increase as temperatures are raised for higher efficiency in advanced engines and it is necessary to evaluate its effect, particularly for first-stage vanes and blades partially subjected to the hot environment of the combustor. Calculations are often made with radiation neglected within the thermal barrier coating. Radiative effects were demonstrated in Ref. 3 where an analytical procedure was developed using the two-flux equations for the radiative contribution. The two-flux method includes, without difficulty, the large scattering characteristic of zirconia. The analysis in Ref. 3 considers two spectral regions with one region opaque, and a shooting method was used to solve the two-flux differential equation. If multiple bands are used, the shooting procedure does not work well for bands with a large optical thickness where effects at the two boundaries become decoupled. This Note provides an improved solution method by deriving a Green's function solution for the two-flux equation. This gives an analytical expression that includes the two-point boundary conditions. An improved convergence procedure was also developed. The analytical expressions are readily evaluated to provide the temperature distribution in a semitransparent coating on a metal wall.

The two-flux equations were shown in Ref. 4 to give accurate results for spectral plane layer composites; hence, the two-flux method is used here to provide a simplified procedure since the exact spectral radiative transfer equations, including large scattering, are rather complicated. The analysis follows from Ref. 5 where a Green's function was derived for a transient analysis of a semitransparent layer subjected to radiation and convection. The derivation is extended here for a thermal barrier coating on a metal wall and the relations are applied for multiple spectral bands.

Analysis

A composite consists of a semitransparent thermal barrier coating on a metal wall (Fig. 1). In the coating the total heat flux is the sum of radiation and conduction and is constant in the x direction. The temperatures in the coating can be found by integrating the energy equation to give the form[3]

$$T(x) = T(0) - \frac{1}{k_c}\left[q_{\rm tot}x - \frac{1}{3}\int_{\lambda=0}^{\infty}\frac{G_\lambda(0) - G_\lambda(x)}{K_\lambda}\,\mathrm{d}\lambda\right] \quad (1)$$

where $T(0)$, $q_{\rm tot}$, and $G_\lambda(x)$ must be determined, as will be described.

The flux quantity $G_\lambda(x)$ is obtained by solving a second-order differential equation[3] that includes the local blackbody emission that will be obtained by iteration:

$$\frac{\mathrm{d}^2G_\lambda(x)}{\mathrm{d}x^2} - 3K_\lambda^2(1-\Omega_\lambda)G_\lambda(x) = -3K_\lambda^2(1-\Omega_\lambda)4n^2e_{\lambda b}(x) \quad (2)$$

Two boundary conditions are required for solving Eq. (2). At $x = 0$ the condition for $G_\lambda(0)$ at the semitransparent boundary of the coating is[3]

$$G_\lambda(0) - \frac{2}{3K_\lambda}\frac{1+\rho_i}{1-\rho_i}\frac{\mathrm{d}G_\lambda}{\mathrm{d}x}\bigg|_{x=0} = 4\frac{1-\rho_o}{1-\rho_i}q_{\lambda r1} \quad (3a)$$

The boundary condition at the coating–metal interface $x = \delta_c$ is[3]

$$G_\lambda(\delta_c) + \frac{2}{3K_\lambda}\left(\frac{2}{\varepsilon_m} - 1\right)\frac{\mathrm{d}G_\lambda}{\mathrm{d}x}\bigg|_{x=\delta_c} = 4n^2e_{\lambda b}(\delta_c) \quad (3b)$$

Equation (2) with the boundary conditions Eqs. (3a) and (3b) is solved by obtaining a Green's function. Following the procedure in Ref. 5, the Green's function was derived as (in dimensionless variables)

$$g_\lambda(X,\xi) = \begin{cases}\left[\dfrac{\sinh m_\lambda(1-\xi) + P_{m\lambda}m_\lambda\cosh m_\lambda(1-\xi)}{m_\lambda\,\mathrm{Denom}_\lambda}\right](\sinh m_\lambda X + P_{i\lambda}m_\lambda\cosh m_\lambda X) & 0 \le X < \xi\\[2ex] \left[\dfrac{\sinh m_\lambda\xi + P_{i\lambda}m_\lambda\cosh m_\lambda\xi}{m_\lambda\,\mathrm{Denom}_\lambda}\right][\sinh m_\lambda(1-X) + P_{m\lambda}m_\lambda\cosh m_\lambda(1-X)] & \xi < X \le 1\end{cases} \quad (4a)$$

where $X = x/\delta_c$, and the denominator in Eq. (4a) is

$$\mathrm{Denom}_\lambda = (1 + P_{i\lambda}P_{m\lambda}m_\lambda^2)\sinh m_\lambda + (P_{i\lambda} + P_{m\lambda})m_\lambda\cosh m_\lambda \quad (4b)$$

The $g_\lambda(X,\xi)$ in Eq. (4a) is used to account for the nonhomogeneous term in Eq. (2) when computing $\tilde{G}_\lambda(X)$. To obtain the complete solution for $\tilde{G}_\lambda(X)$ the solution for the homogeneous part of Eq. (2) is needed. Following the procedure in Ref. 5, this has the solution

$$\tilde{G}_{h\lambda}(X) = A_\lambda\sinh m_\lambda X + B_\lambda\cosh m_\lambda X \quad (5a)$$

where the boundary conditions Eqs. (3a) and (3b) are applied to give relations for B_λ and A_λ:

$$B_\lambda = \frac{4}{\mathrm{Denom}_\lambda}[R_o(\sinh m_\lambda + P_{m\lambda}m_\lambda\cosh m_\lambda)\tilde{q}_{\lambda r1} + P_{i\lambda}m_\lambda n^2\tilde{e}_{\lambda b}(1)] \quad (5b)$$

$$A_\lambda = \frac{B_\lambda - 4R_o\tilde{q}_{\lambda r1}}{P_{i\lambda}m_\lambda} \quad (5c)$$

By adding $\tilde{G}_{h\lambda}(X)$ and the nonhomogeneous solution obtained with $g_\lambda(X)$, the solution of Eq. (2) is

$$\tilde{G}_\lambda(X) = \tilde{G}_{h\lambda}(X) + 4m_\lambda^2n^2\int_0^1 g_\lambda(X,\xi)\tilde{e}_{\lambda b}(\xi)\,\mathrm{d}\xi \quad (6)$$

Now that the solution for $G_\lambda(x)$ has been obtained, relations will be found for $T(0)$ and $q_{\rm tot}$ in Eq. (1). These are obtained by writing total energy flux relations for the coating and metal wall. At the outer boundary of the coating the total heat flux consists of convection, radiation exchange in the opaque spectral region, and radiation within the semitransparent region. The incident spectral radiation flux is $q_{\lambda r1}$. By use of Eq. (10b) in Ref. 3, and Eq. (3a)

$$q_{\rm tot} = h_1[T_{g1} - T(0)] + (1-\rho_o)\int_{\text{opaque bands}}[q_{\lambda r1} - e_{\lambda b}(0)]\,\mathrm{d}\lambda - \frac{1}{2R_i}\int_{\lambda=0}^{\infty}G_\lambda(0)\,\mathrm{d}\lambda + 2\frac{R_o}{R_i}\int_{\text{semitransparent bands}}q_{\lambda r1}\,\mathrm{d}\lambda \quad (7)$$

At the interface between the semitransparent coating and metal, continuity of temperature gives, $T_{\rm coating}(\delta_c) = T_{\rm metal}(\delta_c)$. For the opaque metal wall, heat is transferred only by conduction. At the cooled side of the metal wall, heat flow is by convection to the cooling gas; there is no radiative exchange because the interior of the turbine blade is at fairly uniform temperature and the temperature is relatively low. Hence, inside the blade $q_{\rm tot} = h_2[T(\delta_c + \delta_m) - T_{g2}]$. Then

$$q_{\rm tot} = \left(\frac{\delta_m}{k_m} + \frac{1}{h_2}\right)^{-1}[T(\delta_c) - T_{g2}]$$

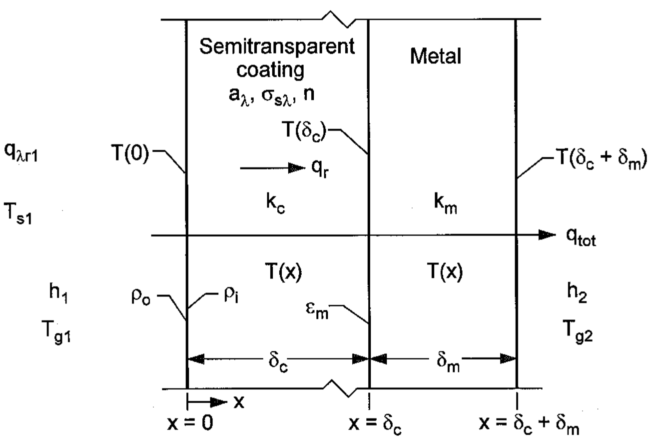

Fig. 1 Geometry and nomenclature for a semitransparent thermal barrier coating on a metal wall, with external radiation and convection on the exposed surface of the coating and convective cooling on the exposed surface of the metal.

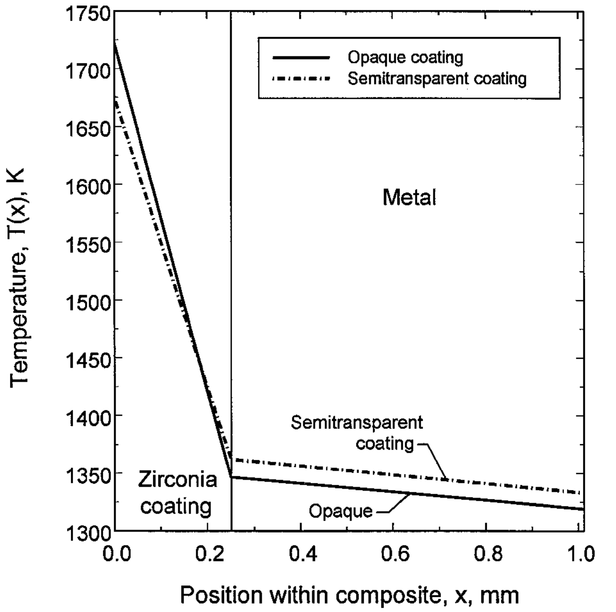

Fig. 2 Turbine first-stage blade wall temperature distributions for a metal wall with a semitransparent thermal barrier coating, compared with an opaque thermal barrier coating. Parameters (units are in Nomenclature): $h_1 = 3014$, $h_2 = 3768$, $k_c = 0.8$, $k_m = 33$, $\delta_c = 0.25 \times 10^{-3}$, $\delta_m = 0.762 \times 10^{-3}$, $n = 1.58$, $T_{s1} = T_{g1} = 2000$, $T_{g2} = 1000$; for semitransparent coating $a_\lambda = 60, 20, 110$ and $\sigma_{s\lambda} = 30{,}000, 7500, 4500$ for $\lambda = 0$ to 1, 1 to 4, 4 to 5 μm, and the coating is opaque for $\lambda > 5$ μm.

The $T(\delta_c)$ is eliminated by using Eq. (1) at $x = \delta_c$; this yields the relation

$$\left(\frac{\delta_c}{k_c} + \frac{\delta_m}{k_m} + \frac{1}{h_2}\right) q_{\text{tot}} = T(0) - T_{g2} + \frac{1}{3k_c}\int_{\lambda=0}^{\infty} \frac{G_\lambda(0) - G_\lambda(\delta_c)}{K_\lambda}\, d\lambda \tag{8}$$

Equations (7) and (8) are solved numerically for $T(0)$ and q_{tot}, and the temperature distribution is then evaluated from Eq. (1). The converged solution for $T(x)$ is found by iteration, as will be described.

Multiple Spectral Band Form of the Radiative Relations

For a zirconia thermal barrier coating the semitransparent region extends to a cutoff wavelength of about 5 μm, beyond which the extinction coefficient becomes quite large[1,6] and the coating is usually assumed opaque. Multiple spectral bands for zirconia or other materials can be used to include spectral property variations in the semitransparent region. The equations are given here for J spectral bands in the region $\lambda = 0$ to λ_{uJ}, and the coating is opaque for $\lambda > \lambda_{uJ}$, which is the range above the upper limit of the Jth band.

To obtain the temperature distribution in the thermal barrier coating, Eq. (1) is summed over the bands in the semitransparent region to give in dimensionless form

$$t(X) = t(0) - \frac{1}{N_c}\left[\tilde{q}_{\text{tot}}X - \frac{1}{3}\sum_{j=1}^{J}\frac{\tilde{G}_j(0) - \tilde{G}_j(X)}{\kappa_j}\right] \tag{9}$$

To obtain $\tilde{G}_j(X)$, Eq. (6) is used for each band in the form

$$\tilde{G}_j(X) = \tilde{G}_{hj}(X) + 4m_j^2 n^2 \int_0^1 g_j(X, \xi)t(\xi)^4[F_{uj}[t(\xi)] - F_{lj}[t(\xi)]]\, d\xi \tag{10}$$

where $\tilde{G}_{hj}$ and g_j are evaluated using the properties in the jth band. $F_{uj}(t)$ is the fraction of blackbody energy for emission at t in the wavelength range $\lambda = 0$ to λ_{uj}, which is at the upper limit of the jth band, and F_{lj} corresponds to the lower limit of λ of the band. $F(t)$ was evaluated from the summation form in Ref. 7. From Eqs. (7) and (8) the equations for $t(0)$ and $\tilde{q}_{\text{tot}}$ are as follows, where it has been specified that $q_{\lambda r1}$ is from blackbody surroundings at T_{s1}:

$$\left(1 + \frac{k_c\delta_m}{k_m\delta_c} + \frac{N_c}{H_2}\right)\Bigg[H_1[1 - t(0)] + (1 - \rho_o) \times \left(t_{s1}^4[1 - F_{uJ}(t_{s1})] - t(0)^4[1 - F_{uJ}[t(0)]]\right) - \frac{1}{2R_i}\sum_{j=1}^{J}\tilde{G}_j(0) + 2\frac{R_o}{R_i}t_{s1}^4F_{uJ}(t_{s1})\Bigg] - N_c[t(0) - t_{g2}] - \frac{1}{3}\sum_{j=1}^{J}\frac{\tilde{G}_j(0) - \tilde{G}_j(1)}{\kappa_j} = 0 \tag{11a}$$

$$\tilde{q}_{\text{tot}} = \left(1 + \frac{k_c\delta_m}{k_m\delta_c} + \frac{N_c}{H_2}\right)^{-1} \times \left\{N_c[t(0) - t_{g2}] + \frac{1}{3}\sum_{j=1}^{J}\frac{\tilde{G}_j(0) - \tilde{G}_j(1)}{\kappa_j}\right\} \tag{11b}$$

A root solver is used to obtain $t(0)$ from Eq. (11a), and $\tilde{q}_{\text{tot}}$ is then found from Eq. (11b).

Solution Method

An iterative solution was used to obtain $t(X)$ and was found to give rapid convergence for all calculations that were made. Less than 10 iterations were usually required to reach convergence to five figures in the dimensionless temperature. The opaque heat conduction solution was used as a first guess for $t(X)$. Then using the radiation properties for each band, the Green's function was evaluated for each band from Eq. (4), and the homogeneous solution from Eq. (5). Using Eq. (10) then gave $\tilde{G}_j(X)$ for each band. Equations (11a) and (11b) are then solved for $t(0)$ and $\tilde{q}_{\text{tot}}$, and Eq. (9) is used to evaluate $t(X)$, which is the temperature distribution to start the next iteration using Eqs. (4) and (5). The procedure is continued until $t(X)$ has converged.

Results and Discussion

An illustrative example is in Fig. 2 for a 0.25-mm-thick zirconia coating on a turbine blade. For an advanced engine, the combustion chamber pressure is high enough that the combined gas and soot radiation are assumed to provide a black environment at T_{s1}. To demonstrate a maximum radiative effect a first-stage blade is considered that is assumed to be exposed to backbody surroundings at T_{s1} and the zirconia surface is assumed to be clean. Radiation properties were approximated from Ref. 1 by three semitransparent bands, with the coating opaque for $\lambda > 5$ μm; numerical values of the parameters are in the figure caption. Temperatures are shown for opaque and semitransparent zirconia coatings. Semitransparency increases the effective thermal conductivity in the coating and this yields higher metal temperatures (about 15 K higher for this example) than for an opaque coating. These results illustrate how the analytical solution developed here can be used to examine the effect of coating semitransparency for various parameters.

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